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General graviton exchange graph for four-point functions in the AdS/CFT correspondence

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Abstract

In this paper, we explicitly compute the graviton exchange graph for scalar fields with arbitrary conformal dimension Δ in arbitrary spacetime dimension d. This results in an analytical function in Δ as well as in d.

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1. Introduction

In the past few years, the Maldacena conjecture [1–3] gave some deep insight into the structure of strongly coupled gauge theories. The possibility of exploiting the strong coupling regime lies in the description of this theory in this regime by a dual theory, which is weakly coupled in some limit and is therefore tractable by perturbation theory. Currently the most investigated version of Maldacena's conjecture is the correspondence between type II B string theory on $AdS_5 \times S^5$ or its tractable limit II B supergravity in the $AdS_5 \times S^5$ background, which appears for $N \to \infty$ and $\alpha' \to 0$ on the one hand and the $N \to \infty$ and $\lambda \to \infty$ limit of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with gauge group $SU(N)$ on the other.

Besides this model, derived from superstring theory, there are two other models derived by Maldacena from considerations of M2/M5 branes in eleven-dimensional M-theory. One obtains a correspondence between eleven-dimensional supergravity in the background of $AdS_{4/7} \times S^{7/4}$ and the three- and six-dimensional conformal field theories on the boundaries of AdS_4 and AdS_7, respectively.

However, as noted in [3] and proved in the context of algebraic quantum field theory in [4], one can also view AdS/CFT as a general feature of quantum field theory, independent of string or M-theory. Appealing to this argument, one can consider field theory models on a $(d+1)$-dimensional AdS space and use the AdS/CFT correspondence to define d-dimensional conformal field theories.

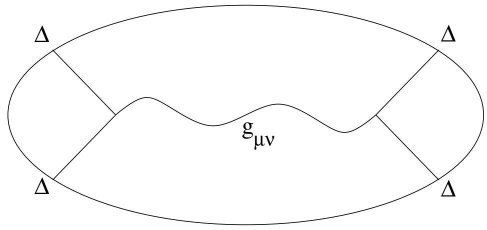

Figure 1. Witten graph of the graviton exchange amplitude.

In conformal field theory, especially in the two-dimensional case, one is used to handling models without any reference to a Lagrangian only by considering correlation functions. There are many results on the topic of correlation functions in AdS/CFT in the literature [5–8], which can be taken independent of string or M-theory. Thus we have a recipe for the definition of conformal field theories, at least perturbatively.

The aim of this paper is the calculation of the 'graviton exchange graph', which describes the interaction of four scalar fields of conformal dimension Δ in d spacetime dimensions via the exchange of a graviton in the bulk of the AdS space. This graph was already explicitly computed in the beginning of AdS/CFT in [10] for the case of $d = \Delta = 4$, because this is the interesting case for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Since the method of [10] and the refinement in [9] seems to work only for integer conformal and spacetime dimensions, we have altered the way of computation to include the cases of general Δ and d.

Our main motivation for the calculation of this general graviton exchange graph is the investigation of structural issues of the AdS/CFT correspondence as a tool for defining conformal field theories. Nevertheless, it has applications in the above-mentioned superconformal field theory derived from M2 branes, because AdS/CFT predicts among others half integers for the dimensions Δ_{CPO} of the chiral primary operators [11].

2. Calculation of the graviton exchange graph

2.1. The problem and the strategy

We want to calculate the amplitude in figure 1. Starting point is the following result taken from [9],

$$G(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \tilde{c} \int \frac{\mathrm{d}^{d+1} w}{w_0^{d+1}} A^{\mu\nu}(w, \vec{x}_1, \vec{x}_3) T_{\mu\nu}(w, \vec{x}_2, \vec{x}_4) \tag{1}$$

where we use the notation of the above reference and $\tilde{c}$ denotes the normalization of the graviton propagator. Here the factor $A^{\mu\nu}$ is calculated and the result is

$$A^{\mu\nu}(w, \vec{x}_1, \vec{x}_3) = |\vec{x}_{13}|^{-2\Delta} \frac{\partial w'^{\lambda}}{\partial w^{\mu}} \frac{\partial w'^{\rho}}{\partial w^{\nu}} I_{\lambda\rho}(w' - \vec{x}'_{13}) \tag{2}$$

with

$$I_{\mu\nu} = g_{\mu\nu} \frac{1}{1-d} \phi(t) + \frac{\delta_{0\mu}\delta_{0\nu}}{w_0^2} \phi(t) + \text{gauge part} \tag{3}$$

where $\vec{x}'_i$ and w' denote coordinates which are obtained from $\vec{x}_i$ and w respectively by an inversion and a succeeding translation by x_1. Moreover, we have defined

$$t = \frac{w_0^2}{w^2} = \frac{w_0^2}{w_0^2 + \vec{w}^2} \tag{4}$$

$$\begin{aligned}\phi(t) &= -\frac{\Delta}{2}\left(\frac{t}{t-1}\right)^{\frac{d}{2}-1}\int_1^t \mathrm{d}t' t'^{\Delta-\frac{d}{2}}(t'-1)^{\frac{d}{2}-2} \\ &= \frac{\Delta}{2}\left(\frac{t}{1-t}\right)^{\frac{d}{2}-1}\left\{\frac{t^{\Delta-\frac{d}{2}+1}}{\Delta-\frac{d}{2}+1}F\left[\begin{matrix}2-\frac{d}{2},\Delta-\frac{d}{2}+1\\ \Delta-\frac{d}{2}+2\end{matrix};t\right]\right. \\ &\quad\left.-\frac{1}{\Delta-\frac{d}{2}+1}F\left[\begin{matrix}2-\frac{d}{2},\Delta-\frac{d}{2}+1\\ \Delta-\frac{d}{2}+2\end{matrix};1\right]\right\}.\end{aligned} \tag{5}$$

As usual the gauge part can be neglected, because it gives no contribution to the integral. Thus we obtain after insertion

$$G(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \tilde{c}(|\vec{x}_{13}||\vec{x}_{12}||\vec{x}_{14}|)^{-2\Delta}\int\frac{\mathrm{d}^{d+1}w'}{w_0'^{d+1}}I^{\mu\nu}(w'-\vec{x}'_{13})T_{\mu\nu}(w',\vec{x}'_{12},\vec{x}'_{14}). \tag{6}$$

Now we write the last integral as $I_1 + I_2$, where I_1 denotes the term containing $g^{\mu\nu}T_{\mu\nu}$ and I_2 the term with T_{00}. Explicit expressions for $g^{\mu\nu}T_{\mu\nu}$ and T_{00} are given in [10],

$$\begin{aligned}g^{\mu\nu}T_{\mu\nu}(w,\vec{x},\vec{y}) &= \left(\frac{1-d}{2}\Box_w - 2m^2\right)\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta} \\ &= ((1-d)(\Delta^2+m^2)-2m^2)\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta} \\ &\quad -2(1-d)\Delta^2|\vec{x}-\vec{y}|^2\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta+1}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta+1}\end{aligned} \tag{7}$$

where we used the defining property of the bulk to boundary propagator with $m^2 = \Delta(\Delta-d)$,

$$\Box_w\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta} = m^2\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta}. \tag{8}$$

From [10] we further learn that

$$\begin{aligned}T_{00}(w,\vec{x},\vec{y}) = \Delta^2&\left\{\frac{1-m^2/\Delta^2}{w_0^2}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta}\right. \\ &-\frac{4}{w_0}\left[\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta+1}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta}\right. \\ &\left.+\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta+1}\right] \\ &\left.+\left(8+\frac{2}{w_0^2}|\vec{x}-\vec{y}|^2\right)\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}|^2}\right)^{\Delta+1}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{y}|^2}\right)^{\Delta+1}\right\}.\end{aligned} \tag{9}$$

Now we insert (7) and (9) into (6), expand the hypergeometric funtion into a series and obtain

$$\begin{aligned}G(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) &= \tilde{c}(|\vec{x}_{13}||\vec{x}_{12}||\vec{x}_{14}|)^{-2\Delta}\frac{\Delta^3}{2\Delta-d+2} \\ &\times\left\{\left(2+\frac{2}{d-1}\frac{m^2}{\Delta^2}\right)\left[\mathbf{M}_{0,0}-\frac{\Gamma\left(\Delta-\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(\Delta)}\overline{\mathcal{M}}_{0,0}\right]\right.\end{aligned}$$

$$
- 4\left[(\mathbf{M}_{1,0}+\mathbf{M}_{0,1}) - \frac{\Gamma\left(\Delta-\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(\Delta)}(\overline{\mathcal{M}}_{1,0}+\overline{\mathcal{M}}_{0,1})\right]
+8\left[\mathbf{M}_{1,1} - \frac{\Gamma\left(\Delta-\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(\Delta)}\overline{\mathcal{M}}_{1,1}\right]\Bigg\} \tag{10}
$$

where we have defined

$$
\mathbf{M}_{s,\tilde{s}} := \sum_{k\geqslant 0}\frac{\left(2-\frac{d}{2}\right)_k\left(\Delta-\frac{d}{2}+1\right)_k}{k!\left(\Delta-\frac{d}{2}+2\right)_k}\mathcal{M}_{k,s,\tilde{s}} \tag{11}
$$

together with the 'standard integrals'

$$
\mathcal{M}_{k,s,\tilde{s}} = \int\frac{\mathrm{d}^{d+1}w}{w_0^{d+1}}\left(\frac{w_0^2}{|\vec{w}-\vec{x}'_{13}|^2}\right)^{\frac{d}{2}-1}\left(\frac{w_0^2}{w_0^2+|\vec{w}-\vec{x}'_{13}|^2}\right)^{\Delta-\frac{d}{2}+1+k} w_0^{s+\tilde{s}}
\times\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}'_{12}|^2}\right)^{\Delta+s}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}'_{14}|^2}\right)^{\Delta+\tilde{s}}
$$

$$
\overline{\mathcal{M}}_{s,\tilde{s}} = \int\frac{\mathrm{d}^{d+1}w}{w_0^{d+1}}\left(\frac{w_0^2}{|\vec{w}-\vec{x}'_{13}|^2}\right)^{\frac{d}{2}-1} w_0^{s+\tilde{s}}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}'_{12}|^2}\right)^{\Delta+s}\left(\frac{w_0}{w_0^2+|\vec{w}-\vec{x}'_{14}|^2}\right)^{\Delta+\tilde{s}}. \tag{12}
$$

Now the remaining problem to be solved is the calculation of these integrals. We will give the solution as a power series in some conformally invariant standard variables defined below (16).

2.2. *Calculation of* $\overline{\mathcal{M}}$

We start with the integral $\overline{\mathcal{M}}$. The denominators are handled with the usual Feynman parameters and we obtain after performing the $(d+1)$-dimensional w-integral

$$
\overline{\mathcal{M}}_{s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma(\Delta+s+\tilde{s}-1)\Gamma(\Delta)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma(\Delta+s)\Gamma(\Delta+\tilde{s})}\int_0^1 \mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\,\delta\left(\sum_{i=2}^4 t_i-1\right)
\times t_2^{\Delta+s-1}t_3^{\frac{d}{2}-2}t_4^{\Delta+\tilde{s}-1}(t_2+t_4)^{-(\Delta+s+\tilde{s}-1)}y^{-\Delta} \tag{13}
$$

where we have defined

$$
y = \sum_{i=2}^4 t_i|\vec{x}'_{1i}|^2 - \left(\sum_{i=2}^4 t_i\vec{x}'_{1i}\right)^2 = \sum_{i<j}t_it_j|\vec{x}'_{1i}-\vec{x}'_{1j}|^2. \tag{14}
$$

Now we use

$$
|\vec{x}'_{1i}-\vec{x}'_{1j}|^2 = \frac{\vec{x}_{ij}^2}{\vec{x}_{1i}^2\vec{x}_{1j}^2} \tag{15}
$$

integrate t_3 (which is trivial), introduce the conformally invariant variables

$$
u := \frac{\vec{x}_{13}^2\vec{x}_{24}^2}{\vec{x}_{12}^2\vec{x}_{34}^2} \qquad v := \frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{12}^2\vec{x}_{34}^2} \tag{16}
$$

and change the integration variables

$$
t_2 = \sigma\rho \qquad t_4 = \sigma(1-\rho). \tag{17}
$$

After that we find

$$y = \sigma \frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2} (1 - (1 - v)\rho) \left(1 - \sigma \left(1 - \frac{\rho(1-\rho)u}{1 - (1-v)\rho}\right)\right) \tag{18}$$

and we can perform the resulting σ-integral to obtain

$$\begin{aligned} \overline{\mathcal{M}}_{s,\tilde{s}} = {} & \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma(\Delta + s + \tilde{s} - 1)\Gamma(\Delta)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2}\right)^{-\Delta} \\ & \times \int \mathrm{d}\rho\, \rho^{\Delta+s-1} (1-\rho)^{\Delta+\tilde{s}-1} (1 - (1-v)\rho)^{-\Delta} F\left[\begin{matrix} \Delta, 1 \\ \frac{d}{2} \end{matrix}; 1 - \frac{\rho(1-\rho)u}{1-(1-v)\rho}\right]. \end{aligned} \tag{19}$$

To calculate the remaining integral, we use an analytic continuation formula for the hypergeometric function to translate the argument of the hypergeometric function by 1 (equation 9.131, 2 of [12]). The two resulting hypergeometric functions can then be expanded as a power series and the ρ-integral can be carried out, again in terms of Gaussian hypergeometric functions. The result is

$$\begin{aligned} \overline{\mathcal{M}}_{s,\tilde{s}} = {} & \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma(\Delta + s + \tilde{s} - 1)\Gamma(\Delta)}{\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2}\right)^{-\Delta} \left\{ \frac{\Gamma\left(\frac{d}{2} - \Delta - 1\right)}{\Gamma\left(\frac{d}{2} - \Delta\right)\Gamma\left(\frac{d}{2} - 1\right)} \right. \\ & \times \sum_{l=0}^{\infty} \frac{B(\Delta + s + l, \Delta + \tilde{s} + l)(\Delta)_l}{\left(\Delta - \frac{d}{2} + 2\right)_l} u^l F\left[\begin{matrix} \Delta + l, \Delta + s + l \\ 2\Delta + s + \tilde{s} + 2l \end{matrix}; 1 - v\right] \\ & + \frac{\Gamma\left(\Delta + 1 - \frac{d}{2}\right)}{\Gamma(\Delta)} \sum_{l=0}^{\infty} \frac{B\left(\frac{d}{2} + s + l - 1, \frac{d}{2} + \tilde{s} + l - 1\right)\left(\frac{d}{2} - 1\right)_l}{l!} \\ & \left. \times u^{l + \frac{d}{2} - \Delta - 1} F\left[\begin{matrix} \frac{d}{2} - 1 + l, \frac{d}{2} + s + l - 1 \\ d + s + \tilde{s} + 2l - 2 \end{matrix}; 1 - v\right] \right\}. \end{aligned} \tag{20}$$

2.3. Calculation of M

Now that we have seen how to compute $\overline{\mathcal{M}}$, the procedure for $\mathcal{M}$ is straightforward. As before, we introduce Feynman parameters (here we need four of them), perform the w-integral and integrate (trivially) over t_1 to obtain

$$\begin{aligned} \mathcal{M}_{k,s,\tilde{s}} = {} & \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(2\Delta + k + s + \tilde{s} - \frac{d}{2}\right)\Gamma(\Delta)}{\Gamma\left(\frac{d}{2} - 1\right)\Gamma\left(\Delta - \frac{d}{2} + 1 + k\right)\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \int \mathrm{d}t_2\, \mathrm{d}t_3\, \mathrm{d}t_4\, t_2^{\Delta+s-1} t_3^{\Delta+k-\frac{d}{2}} t_4^{\Delta+\tilde{s}-1} \\ & \times (1 - t_2 - t_3 - t_4)^{\frac{d}{2}-2} (t_2 + t_3 + t_4)^{-(2\Delta+k+s+\tilde{s}-\frac{d}{2})} y^{-\Delta} \end{aligned} \tag{21}$$

where we used the abbreviation y given above. Again we make the substitution (17) and obtain for the integral over t_3

$$\begin{aligned} \int_0^{1-\sigma} & \mathrm{d}t_3\, t_3^{\Delta - \frac{d}{2} + k} (1 - \sigma - t_3)^{\frac{d}{2}-2} (\sigma + t_3)^{-(2\Delta+k+s+\tilde{s}-\frac{d}{2})} = (1-\sigma)^{\Delta+k-1} \sigma^{-(2\Delta+k+s+\tilde{s}-\frac{d}{2})} \\ & \times B\left(\Delta + k + 1 - \frac{d}{2}, \frac{d}{2} - 1\right) F\left[\begin{matrix} 2\Delta + k + s + \tilde{s} - \frac{d}{2}, \Delta + k - \frac{d}{2} + 1 \\ \Delta + k \end{matrix}; \frac{\sigma - 1}{\sigma}\right] \\ & = (1-\sigma)^{\Delta+k-1} B\left(\Delta + k + 1 - \frac{d}{2}, \frac{d}{2} - 1\right) \\ & \times F\left[\begin{matrix} 2\Delta + k + s + \tilde{s} - \frac{d}{2}, \frac{d}{2} - 1 \\ \Delta + k \end{matrix}; \frac{\sigma - 1}{\sigma}\right] \end{aligned} \tag{22}$$

where we used the first transformation formula of equations 9.131, 1 in [12]. We plug (22) back into (21), expand the hypergeometric function and integrate over σ to obtain

$$\mathcal{M}_{k,s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(2\Delta + k + s + \tilde{s} - \frac{d}{2}\right)\Gamma(\Delta)}{\Gamma(\Delta + k)\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2}\right)^{-\Delta}$$
$$\times \sum_{n\geqslant 0} \frac{\left(2\Delta + k + s + \tilde{s} - \frac{d}{2}\right)_n \left(\frac{d}{2} - 1\right)_n}{n!(\Delta + k)_n} B(\Delta + s + \tilde{s}, \Delta + k + n)$$
$$\times \int \mathrm{d}\rho\, \rho^{\Delta+s-1}(1-\rho)^{\Delta+\tilde{s}-1}(1-(1-v)\rho)^{-\Delta}$$
$$\times F\left[\begin{matrix}\Delta, \Delta + s + \tilde{s} \\ 2\Delta + k + n + s + \tilde{s}\end{matrix}; 1 - \frac{\rho(1-\rho)u}{1-(1-v)\rho}\right]. \tag{23}$$

Again, we apply the analytic continuation formula 9.131, 2 of [12] to translate the argument of the last hypergeometric function by 1. Since we run into poles in the formulae[1], we make the shift $k \mapsto k + \varepsilon$ and obtain after performing the last integration

$$\mathcal{M}_{k,s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(2\Delta + k + \varepsilon + s + \tilde{s} - \frac{d}{2}\right)}{\Gamma(\Delta + k + \varepsilon)\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2}\right)^{-\Delta}$$
$$\times \sum_{n\geqslant 0} \frac{\left(2\Delta + k + \varepsilon + s + \tilde{s} - \frac{d}{2}\right)_n \left(\frac{d}{2} - 1\right)_n}{n!(\Delta + k + \varepsilon)_n \Gamma(\Delta + k + \varepsilon + n + s + \tilde{s})}$$
$$\times \left\{ \Gamma(k + \varepsilon + n)\Gamma(1 - k - \varepsilon - n) \sum_{m,l\geqslant 0} \frac{u^l (1-v)^m}{l!m!} \right.$$
$$\times \frac{\Gamma(\Delta + \tilde{s} + l)\Gamma(\Delta + s + \tilde{s} + l)}{\Gamma(1 - k - \varepsilon - n + l)} \frac{\Gamma(\Delta + l + m)\Gamma(\Delta + s + l + m)}{\Gamma(2(\Delta + l) + s + \tilde{s} + m)}$$
$$+ \Gamma(-k - \varepsilon - n)\Gamma(1 + k + \varepsilon + n) \sum_{m,l\geqslant 0} \frac{u^{k+\varepsilon+n+l}(1-v)^m}{l!m!}$$
$$\times \frac{\Gamma(\Delta + \tilde{s} + k + \varepsilon + n + l)\Gamma(\Delta + s + \tilde{s} + k + \varepsilon + n + l)}{\Gamma(1 + k + \varepsilon + n + l)}$$
$$\left. \times \frac{\Gamma(\Delta + k + \varepsilon + n + l + m)\Gamma(\Delta + s + k + \varepsilon + n + l + m)}{\Gamma(2(\Delta + k + \varepsilon + n + l) + s + \tilde{s} + m)} \right\}. \tag{24}$$

The next step is the summation over k. We observe that up to two factors all k-summands in $\mathcal{M}_{k,s,\tilde{s}}$ solely depend on $k + n$, therefore by introducing the new summation variable $N := k + n$, we shift the summation variables from (k, n) to (k, N). Since the resulting k-sum is a ${}_3F_2$-series, which can be summed by Saalschütz theorem [13], we obtain

$$\mathbf{M}_{s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2} \frac{1}{\Gamma(\Delta + s)\Gamma(\Delta + \tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2}\right)^{-\Delta}$$
$$\times \sum_{N\geqslant 0} \frac{\Gamma\left(2\Delta + s + \tilde{s} - \frac{d}{2} + N + \varepsilon\right)}{\Gamma(\Delta + N + \varepsilon)\Gamma(\Delta + s + \tilde{s} + N + \varepsilon)} \frac{(\Delta)_N}{\left(\Delta - \frac{d}{2} + 2\right)_N} \frac{\pi}{\sin \pi(N + \varepsilon)}$$
$$\times \sum_{l,m\geqslant 0} \frac{u^l(1-v)^m}{l!m!} \left\{ \frac{\Gamma(\Delta + s + \tilde{s} + l)\Gamma(\Delta + \tilde{s} + l)}{\Gamma(1 - N - \varepsilon + l)} \right.$$

[1] These singularities are only formal artefacts, the content is fully analytical of course.

$$\times \frac{\Gamma(\Delta+l+m)\Gamma(\Delta+s+l+m)}{\Gamma(2(\Delta+l)+s+\tilde{s}+m)}$$
$$- u^{N+\varepsilon} \frac{\Gamma(\Delta+s+\tilde{s}+N+\varepsilon+l)\Gamma(\Delta+\tilde{s}+N+\varepsilon+l)}{\Gamma(1+N+\varepsilon+l)}$$
$$\left. \times \frac{\Gamma(\Delta+N+\varepsilon+l+m)\Gamma(\Delta+s+N+\varepsilon+l+m)}{\Gamma(2(\Delta+N+\varepsilon+l)+s+\tilde{s}+m)} \right\}. \tag{25}$$

To perform the sum over N we note that the first double series in u, $1-v$ contains only one N-dependent term

$$\frac{1}{\Gamma(1-N-\varepsilon+l)} = (-1)^N \frac{(\varepsilon-l)_N}{\Gamma(1+l-\varepsilon)} \tag{26}$$

and in the second double series the summands only depend upon $L := N+l$, up to

$$\frac{1}{l!} = (-1)^N \frac{(-L)_N}{L!} \tag{27}$$

so that we obtain after replacing $L \mapsto l$

$$\mathbf{M}_{s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2} \frac{1}{\Gamma(\Delta+s)\Gamma(\Delta+\tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2} \right)^{-\Delta}$$
$$\times \sum_{N\geqslant 0} \frac{\Gamma\big(2\Delta+s+\tilde{s}-\frac{d}{2}+N+\varepsilon\big)}{\Gamma(\Delta+N+\varepsilon)\Gamma(\Delta+s+\tilde{s}+N+\varepsilon)} \frac{(\Delta)_N}{\big(\Delta-\frac{d}{2}+2\big)_N} \frac{\pi}{\sin\pi\varepsilon}$$
$$\times \sum_{l,m\geqslant 0} \frac{u^l(1-v)^m}{l!m!} \left\{ \frac{\Gamma(\Delta+s+\tilde{s}+l)\Gamma(\Delta+\tilde{s}+l)}{\Gamma(1-\varepsilon+l)} (\varepsilon-l)_N \right.$$
$$\times \frac{\Gamma(\Delta+l+m)\Gamma(\Delta+s+l+m)}{\Gamma(2(\Delta+l)+s+\tilde{s}+m)}$$
$$- u^{\varepsilon} \frac{\Gamma(\Delta+s+\tilde{s}+\varepsilon+l)\Gamma(\Delta+\tilde{s}+\varepsilon+l)}{\Gamma(1+\varepsilon+l)} (-l)_N$$
$$\left. \times \frac{\Gamma(\Delta+\varepsilon+l+m)\Gamma(\Delta+s+\varepsilon+l+m)}{\Gamma(2(\Delta+\varepsilon+l)+s+\tilde{s}+m)} \right\}. \tag{28}$$

At last we have to perform the limit $\varepsilon \to 0$. To do this, we observe that the pole $\sin^{-1}\pi\varepsilon$ is cancelled by the zero of $\{\cdots\}$ in (28) at $\varepsilon = 0$, such that the result is analytical, as promised before. We take the limit with the help of l'Hôpital's rule and arrive finally after a short calculation at

$$\mathbf{M}_{s,\tilde{s}} = \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\big(2\Delta+s+\tilde{s}-\frac{d}{2}\big)}{\Gamma(\Delta)\Gamma(\Delta+s)\Gamma(\Delta+\tilde{s})\Gamma(\Delta+s+\tilde{s})} \left(\frac{\vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{14}^2} \right)^{-\Delta}$$
$$\times \sum_{l,m\geqslant 0} \frac{u^l(1-v)^m}{l!m!} \frac{\Gamma(\Delta+s+\tilde{s}+l)\Gamma(\Delta+\tilde{s}+l)}{\Gamma(1+l)}$$
$$\times \frac{\Gamma(\Delta+l+m)\Gamma(\Delta+s+l+m)}{\Gamma(2(\Delta+l)+s+\tilde{s}+m)} \left\{ {}_3F_2 \left[\begin{matrix} 2\Delta+s+\tilde{s}-\frac{d}{2}, 1, -l \\ \Delta+s+\tilde{s}, \Delta-\frac{d}{2}+2 \end{matrix} ; 1 \right] \right.$$
$$\times (-\log u + 2\psi(1+l) - \psi(\Delta+\tilde{s}+l) - \psi(\Delta+s+\tilde{s}+l) - \psi(\Delta+l+m)$$
$$- \psi(\Delta+s+l+m) + 2\psi(2(\Delta+l)+s+\tilde{s}+m))$$
$$+ \sum_{N=1}^{l} \frac{\big(2\Delta+s+\tilde{s}-\frac{d}{2}\big)_N (-l)_N}{(\Delta+s+\tilde{s})_N \big(\Delta-\frac{d}{2}+2\big)_N} \sum_{p=0}^{N-1} \frac{1}{p-l}$$

$$
+(-1)^l l! \frac{\left(2\Delta + s + \tilde{s} - \frac{d}{2}\right)_{l+1}}{(\Delta + s + \tilde{s})_{l+1}\left(\Delta - \frac{d}{2} + 2\right)_{l+1}}
\times {}_3F_2 \left[\begin{matrix} 2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1, 1, 1 \\ \Delta + s + \tilde{s} + l + 1, \Delta - \frac{d}{2} + l + 3 \end{matrix} ; 1 \right] \Bigg\} . \tag{29}
$$

Now we only have to transform the last generalized hypergeometric function in (29) into a finite sum. This can be done in the following way: we first apply the fundamental two-term relation (equation (4.3.1) in [13]) to obtain

$$
{}_3F_2 \left[\begin{matrix} 2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1, 1, 1 \\ \Delta + s + \tilde{s} + l + 1, \Delta - \frac{d}{2} + l + 3 \end{matrix} ; 1 \right] = \frac{\Gamma(\Delta + s + \tilde{s} + l + 1)\Gamma\left(\Delta - \frac{d}{2} + l + 3\right)\Gamma(l+1)}{\Gamma\left(2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1\right)\Gamma(l+2)^2}
\times {}_3F_2 \left[\begin{matrix} \frac{d}{2} - \Delta, l+1, 2 - s - \tilde{s} - \Delta \\ l+2, l+2 \end{matrix} ; 1 \right] . \tag{30}
$$

One immediately recognizes that this series terminates in case of $\Delta \in \mathbb{N}_0$ or $\Delta - \frac{d}{2} \in \mathbb{N}_0$. But we want to have a calculable exact result for any field dimension Δ and any spacetime dimension d, thus we now apply the fundamental three-term relation (eqation (4.3.2.1) in [13]) to (30), where we choose

$$
F_n(2) = F_n(2; 1, 3) \qquad F_n(3) = F_n(3; 4, 5) \tag{31}
$$

in the notation of [13]. We include the pre-factor of the ${}_3F_2$-function in the last summand of (29) and obtain

$$
(-1)^l l! \frac{\left(2\Delta + s + \tilde{s} - \frac{d}{2}\right)_{l+1}}{(\Delta + s + \tilde{s})_{l+1}\left(\Delta - \frac{d}{2} + 2\right)_{l+1}} {}_3F_2 \left[\begin{matrix} 2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1, 1, 1 \\ \Delta + s + \tilde{s} + l + 1, \Delta - \frac{d}{2} + l + 3 \end{matrix} ; 1 \right]
$$
$$
= \frac{l!\Gamma\left(2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1\right)\Gamma(\Delta + s + \tilde{s})\Gamma(\Delta + s + \tilde{s} - 1)}{\Gamma\left(2\Delta + s + \tilde{s} - \frac{d}{2}\right)\Gamma(\Delta + s + \tilde{s} + l)^2\Gamma\left(\Delta - \frac{d}{2} + 2\right)}
\times \Gamma\left(\Delta - \frac{d}{2} + 1\right) {}_3F_2 \left[\begin{matrix} 2\Delta + s + \tilde{s} - \frac{d}{2} + l + 1, -l, \Delta - \frac{d}{2} + 1 \\ \Delta - \frac{d}{2} + 2, \Delta - \frac{d}{2} + 2 \end{matrix} ; 1 \right]
$$
$$
- \frac{(l!)^2\Gamma(\Delta + s + \tilde{s})\Gamma(\Delta + s + \tilde{s} - 1)\Gamma\left(\Delta - \frac{d}{2} + 2\right)\Gamma\left(\Delta - \frac{d}{2} + 1\right)}{\Gamma\left(2\Delta + s + \tilde{s} - \frac{d}{2}\right)\Gamma(\Delta + s + \tilde{s} + l)\Gamma\left(\Delta - \frac{d}{2} + 2 + l\right)} . \tag{32}
$$

Finally, we have got rid of all infinite series for the coefficients and we can calculate for a given spacetime dimension $d > 2$ and a given conformal dimension $\Delta > \frac{d}{2} - 1$ any coefficient of the power series in u and $1 - v$ exactly in finitely many steps.

3. Comparison with previous results

In this section we compare our results with those of [15], where the results of [10] are formulated in the same variables as we use. In the case of the dilaton axion exchange graph $d = \Delta = 4$ one can check that the coefficients of the power series in u and $1 - v$ indeed agree, if we take care of the normalizations of the graviton propagator and the bulk-to-boundary propagators of the scalar fields.

In [15] it was pointed out that the singular power terms in the $(u, 1 - v)$-expansion of the graviton exchange amplitude arise from the exchange of an energy–momentum tensor in the conformal field theory interpretation. Let us see if the same applies to the case of arbitrary d and Δ:

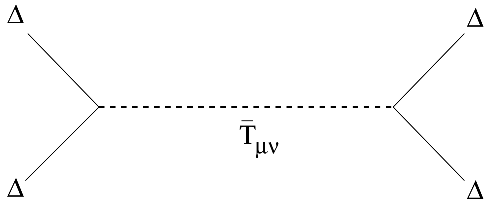

Figure 2. Conformal graph of the energy–momentum exchange amplitude.

The exchange graph for the energy–momentum tensor $\bar{T}_{\mu\nu}$ can be calculated with the help of the 'master formula' of [14]. With that formula we first give the exchange amplitude $W_\delta(x_1, \ldots, x_4; \Delta)$ for a second rank tensor of arbitrary conformal dimension δ with scalars of dimension Δ as exterior legs:

$$\begin{aligned} W_\delta(x_1, \ldots, x_4; \Delta) &= \pi^{\frac{d}{2}} \left(x_{12}^2 x_{34}^2\right)^{-\Delta} C_{\Delta,\Delta} \tilde{C}_{\Delta,\Delta} \Gamma\left(\frac{d}{2} - \delta\right) \Gamma\left(1 - \frac{d}{2} + \delta\right) u^{\frac{\delta}{2}-1-\Delta} \\ &\quad \times \sum_{n\geqslant 0} \frac{u^n}{n!} \frac{1}{\Gamma(\delta+2n)\Gamma\left(\delta - \frac{d}{2} + 1 + n\right)} \left\{ \frac{1}{2}\left(1 - \frac{2u}{d}\right) A_n(0,0)\phi_n(0,0;1-v) \right. \\ &\quad - [A_n(2,0) + A_n(0,2)]\phi_n(2,0;1-v) \\ &\quad \left. + \frac{1}{2} A_n(2,2)[\phi_n(2,2;1-v) + \phi_n(2,-2;1-v)] \right\} \end{aligned} \tag{33}$$

where

$$A_n(i,j) := \left(\frac{\delta+i}{2}\right)_n \left(\frac{\delta-i}{2}\right)_n \frac{\Gamma\left(\frac{\delta+j}{2}+n\right)\Gamma\left(\frac{\delta-j}{2}+n\right)}{\Gamma\left(\frac{d-\delta-j}{2}\right)\Gamma\left(\frac{d-\delta+j}{2}\right)} \tag{34}$$

and

$$\phi_n(i,j;z) := F\left[\begin{matrix} \frac{\delta-i}{2}+n, \frac{\delta-j}{2}+n \\ \delta+2n \end{matrix} ; z\right]. \tag{35}$$

At the dimension of the energy–momentum tensor $\delta = d$ we recognize that $A_n(i,0)$ has a double zero, whereas $A_n(i,2)$ has a simple one. We renormalize the coupling constants $C_{\Delta,\Delta}$, $\tilde{C}_{\Delta,\Delta}$ by absorbing the simple zero:

$$\begin{aligned} C^{(r)}_{\Delta,\Delta} &= \Gamma\left(\frac{d-\delta-2}{2}\right)^{\frac{1}{2}} C_{\Delta,\Delta} \\ \tilde{C}^{(r)}_{\Delta,\Delta} &= \Gamma\left(\frac{d-\delta-2}{2}\right)^{\frac{1}{2}} \tilde{C}_{\Delta,\Delta}. \end{aligned} \tag{36}$$

Then only the terms with $A_n(i,2)$ survive the limit $\delta \to d$. and we get for the energy–momentum tensor exchange amplitude

$$\begin{aligned} W^{(r)}_d(x_1, \ldots, x_4; \Delta) &= \pi^{\frac{d}{2}} \left(x_{12}^2 x_{34}^2\right)^{-\Delta} C^{(r)}_{\Delta,\Delta} \tilde{C}^{(r)}_{\Delta,\Delta} \frac{\Gamma\left(-\frac{d}{2}\right)}{2\,\Gamma\left(\frac{d}{2}\right)} u^{\frac{d}{2}-1-\Delta} \\ &\quad \times \sum_{n\geqslant 0} \frac{u^n}{n!} \frac{\Gamma\left(\frac{d}{2}-1+n\right)\Gamma\left(\frac{d}{2}+n\right)^2}{\Gamma(d+2n)} \left\{ -d\phi_n(0,2) \right. \\ &\quad \left. + \frac{\left(\frac{d}{2}+n\right)\left(\frac{d}{2}-1\right)}{\left(\frac{d}{2}-1+n\right)} (\phi_n(2,2) + \phi_n(2,-2)) \right\}. \end{aligned} \tag{37}$$

Note that a further renormalization is necessary, if d is an even positive integer.

The singular terms in the graviton exchange graph all stem from the second series in $\overline{\mathcal{M}}_{s,\bar{s}}$ (20). We insert the complete second series into (10) and obtain after using some recursion relations for the Gaussian hypergeometric functions

$$G(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)|_{\text{singular part}} = \pi^{\frac{d}{2}} \tilde{c} K_\Delta^4 \left(x_{12}^2 x_{34}^2\right)^{-\Delta} \frac{\Delta^2 \Gamma\left(\Delta - \frac{d}{2} + 1\right)^2}{\Gamma(\Delta)^2 (d-1)} u^{\frac{d}{2}-1-\Delta} \sum_{l \geqslant 0} \cdots \tag{38}$$

where the dots denote the same summands as in (37). Moreover, we installed the normalizations of the bulk-to-boundary propagators K_Δ [5].

Thus we conclude that the second series in (20), which contains the singular terms, exactly coincides with the conformal energy–momentum exchange graph if we set for the coupling constant (taking into account the above-mentioned normalizing factor)

$$C_{\Delta,\Delta}^{(r)} \tilde{C}_{\Delta,\Delta}^{(r)} = \tilde{c} K_\Delta^4 \frac{\Delta^2 \Gamma\left(\Delta - \frac{d}{2} + 1\right)^2}{\Gamma(\Delta)^2} \frac{2\,\Gamma\left(\frac{d}{2}\right)}{(d-1)\Gamma\left(-\frac{d}{2}\right)}. \tag{39}$$

We close by mentioning that the result we have obtained is fully analytical in the conformal dimensions Δ of the scalar fields as well as in the spacetime dimension d. This is a general feature of conformal field theories. For example, the $O(N)$-sigma model has a critical point for $2 < d < 4$ and the amplitudes are analytical in d in this domain. Recently, a conjecture was formulated about the holographic dual of this model in AdS [16]. Thus, the amplitudes calculated by the AdS/CFT procedure in this model are also supposed to show analytical behaviour in this domain.

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